On a property of approximate derivatives

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Introduction

An interesting property of the derivative f' of a continuous function f is that for every open interval (α, β) the inverse $f'^{-1}(\alpha, \beta)$ of (α, β) under f' is either void or of positive measure [1, 2]. This property of the derivative is also shared by the approximate derivative f'_{ap} [6, 11] and the nth Peano derivative f_n [8, 11]. Recently a necessary and sufficient condition is obtained under which a function of Baire class 1 will have the above property [7]. WEIL [12] proved that if f'_{ap} (resp. f_n) exists finitely and if the inverse $f'^{-1}_{ap}(\alpha, \beta)$ (resp. $f_n^{-1}(\alpha, \beta)$) is non-empty then the set $f'^{-1}(\alpha, \beta)$ (resp. $f^{(n)-1}(\alpha, \beta)$, $f^{(n)}$ being the nth derivative of f) is also of positive measure on the sets where it exists. It is also known that if f'_{ap} (resp. f_n) exists at each point then f' (resp. $f^{(n)}$) also exists on an everywhere dense set of intervals [3] (resp. [8]). The purpose of the present note is to establish certain results which will imply the above results as well as the results of WEIL [12].

Definitions and notations

We shall follow the standard definition of approximate derivatives (see [9]). For the definition of nth Peano derivative we refer to [8].

Throughout, all functions considered are real valued defined on the real line. To save space we shall use the following notations:

$$E(*, \alpha) = \{x : f'_{ap}(x) \leqq \alpha\}, \quad E(\beta, *) = \{x : f'_{ap}(x) \geqq \beta\},$$

$$E_0(*, \alpha) = \{x : f'(x) \text{ exists and } f'(x) \leqq \alpha\}, \; E_0(\beta, *) = \{x : f'(x) \text{ exists and } f'(x) \geqq \beta\}.$$

Main results

Theorem 1. *Let f be approximately continuous and let the approximate derivative f'_{ap} — finite or infinite — exits at each point. If for any two reals α, β, $\alpha < \beta$, ($\pm\infty$ admitted), the set*

$$E = \{x : \alpha < f'_{ap}(x) < \beta\}$$

is non-empty, then for every interval I where $I \cap E \neq 0$, there is a sub-interval $J \subset I$ such that

$$J \cap E \neq 0, \quad J \cap E = J \cap E_0,$$

where

$$E_0 = \{\varkappa : f'(\varkappa) \text{ exist and } \alpha < f'(\varkappa) < \beta\}.$$

Proof. We prove the theorem in two steps. In the first step we show that if for an interval I, $I \cap E \neq 0$, then $I \cap E_0 \neq 0$. In the second step we complete the proof.

Step I. Suppose the contrary. Then there is an interval I such that

$$I \cap E \neq 0, \quad I \cap E_0 = 0. \tag{1}$$

Let $\varkappa' \in I \cap E$. Then $\alpha < f'_{ap}(\varkappa') < \beta$. Choose $\alpha < \alpha_1 < f'_{ap}(\varkappa') < \beta_1 < \beta$. Then α_1, β_1 are finite. Let $\{Q\}_1$ and $\{Q\}_2$ be the collection of all non-degenerate components of $I \cap E_0(*, \alpha_1)$ and $I \cap E_0(\beta_1, *)$ respectively. Let $Q \in \{Q\}_1$. Then Q is an interval. Also $Q \subset E(*, \alpha_1)$. Since f'_{ap} possesses Darboux property [5], $\bar{Q} \subset E(*, \alpha_1)$, where $\bar{Q}$ is the closure of Q relative to I. Thus $f'_{ap}(\varkappa) \leqq \alpha_1$ for all $\varkappa \in \bar{Q}$ and hence $f'(\varkappa)$ exists and $f'(\varkappa) \leqq \alpha_1$ for all $\varkappa \in \bar{Q}$ [4]. Since Q is a component, $Q = \bar{Q}$. Thus every member of $\{Q\}_1$ is an interval which is closed relative to I. Similarly every member of $\{Q\}_2$ is also an interval which is closed relative to I.

Let $\{Q\} = \{Q\}_1 \cup \{Q\}_2$. Let $P = I - \bigcup_{Q \in \{Q\}} Q^0$, where Q^0 denotes the interior of Q relative to I. Then P is non-void, since $\varkappa' \in P$. Clearly two distinct members of $\{Q\}$ cannot have a common end point. Hence the set P is perfect in I. We shall show that under the hypothesis f'_{ap}/P has no point of continuity in P. Since f'_{ap} is a function of Baire class 1 [10], it will lead to a contradiction.

Let $\varkappa_0 \in P$ and let J be any open interval containing $\varkappa_0$. Then $J \cap I \cap E(*, \alpha_1)$ and $J \cap I \cap E(\beta_1, *)$ are non-void. For, if $J \cap I \cap E(*, \alpha_1) = 0$, then $f'_{ap}(\varkappa) > \alpha_1$ for all $\varkappa \in J \cap I$ and hence $f'(\varkappa)$ exists and $f'(\varkappa) > \alpha_1$ for all $\varkappa \in J \cap I$. Since from (1) $I \cap E_0 = 0$, we conclude $f'(\varkappa) \geqq \beta_1$ for all $\varkappa \subset J \cap I$ and hence $J \cap I \subset Q^0$ for some $Q \in \{Q\}_2$, which is contrary to the fact that $\varkappa_0 \in J \cap I$ and $\varkappa_0 \in P$. Similar arguments hold for $J \cap I \cap E(\beta_1, *)$. From the above conclusion we assert that $J \cap P \cap E(*, \alpha_1)$ and $J \cap P \cap E(\beta_1, *)$ are also non-void. For, let $\xi \in J \cap I \cap E(*, \alpha_1)$. If $\xi \in P$ then $\xi \in J \cap P \cap E(*, \alpha_1)$ and the assertion follows. If $\xi \notin P$ then $\xi \in Q^0$ for some $Q \in \{Q\}_1$. Let η be the end point of Q which lies between $\varkappa_0$ and ξ. Then $\eta \in J \cap P \cap E(*, \alpha_1)$. Similar arguments hold for $J \cap P \cap E(\beta_1, *)$. Hence

$$\inf_{\varkappa \in J \cap P} f'_{ap}(\varkappa) \leqq \alpha_1, \quad \sup_{\varkappa \in J \cap P} f'_{ap}(\varkappa) \geqq \beta_1$$

showing that the saltus of the function f'_{ap}/P at $\varkappa_0$ is at least $\beta_1 - \alpha_1$. Hence f'_{ap}/P cannot be continuous at $\varkappa_0$. Since $\varkappa_0 \in P$ is arbitrary, this completes the first step of our proof.

Step II. If possible, suppose that there is an interval I such that $I\cap E\neq 0$ but for every sub-interval $J\subset I$ satisfying $J\cap E\neq 0$ the relation

$$I\cap E\neq J\cap E_0 \tag{2}$$

holds. Let $F=I\cap E_0$. Then by Step I F is non-void. Also F is dense in itself. For, if $x_0\in F$ and G is any open interval containing x_0 then since f'_{ap} has Darboux property, there is $x'\neq x_0$ such that $x'\in G\cap I\cap E$. If G_1 is any open interval contained in G and containing x' but not x_0 then $G_1\cap I\cap E\neq 0$ and hence by Step I $G_1\cap I\cap \cap E_0\neq 0$. So, there is $x''\in F$, $x''\neq x_0$ and $x''\in G$. Thus x_0 is a limit point of F. Let $\bar F$ denote the closure of F relative to I. Since F is dense in itself, $\bar F$ is perfect in I. We shall show that f'_{ap} has no point of continuity in $\bar F$ relative to $\bar F$.

Let $x_0\in \bar F$ and let H be any open interval containing x_0. Then there is $x'\in H\cap F$. Hence $x'\in H\cap I$ and $\alpha<f'(x')<\beta$. Choose $\alpha<\alpha_1<f'(x')<\beta_1<\beta$. Then the sets $H\cap I\cap E(*,\alpha_1)$ and $H\cap I\cap E(\beta_1,*)$ are non-void. For, if $H\cap I\cap E(*,\alpha_1)=0$ then $f'_{ap}(x)>\alpha_1$ for all $x\in H\cap I$ and hence $f'(x)$ exists for all $x\in H\cap I$. So, $H\cap I\cap E_0=H\cap I\cap E$ and $x'\in H\cap I\cap E$, which is contrary to (2). Similar arguments hold for $H\cap I\cap E(\beta_1,*)$.

Now for arbitrary $k_1,k_2,\alpha_1<k_1<k_2<\beta_1$, the sets $H\cap F\cap E(*,k_1)$ and $H\cap F\cap E(k_2,*)$ are non-empty. For, by the Darboux property of f'_{ap} there is $\xi\in H\cap I$ such that $\alpha_1<f'_{ap}(\xi)<k_1$. Hence as in Step I, there is $\xi_0\in H\cap I$ such that $f'(\xi_0)$ exists and $\alpha_1<f'(\xi_0)<k_1$ and so $\xi_0\in H\cap F$ showing that the set $H\cap F\cap \cap E(*,k_1)$ is non-empty. Similarly for the set $H\cap F\cap E(k_2,*)$. Hence $H\cap\bar F\cap E(*,k_1)$ and $H\cap \bar F\cap E(k_2,*)$ are also non-empty. Since k_1 and k_2 can be taken very near to α_1 and β_1 respectively, we conclude

$$\inf_{x\in H\cap F} f'_{ap}(x)\leqq \alpha_1,\qquad \sup_{x\in H\cap F} f'_{ap}(x)\geqq \beta_1$$

which shows that the saltus of $f'_{ap}/\bar F$ at x_0 is at least $\beta_1-\alpha_1$. Hence $f'_{ap}/\bar F$ is not continuous at x_0. Since x_0 is an arbitrary point of $\bar F$, it follows that $f'_{ap}/\bar F$ has no point of continuity in $\bar F$. Since f'_{ap} is a function of Baire class I, it provides with a contradiction. This completes the proof.

Putting $\alpha=-\infty$, $\beta=\infty$ we deduce the following known corollary.

Corollary [3]. *Let f have a finite approximate derivative f'_{ap} everywhere. Let $E=\{x\colon f'(x)$ exists$\}$. Then for every interval I, $I\cap E$ contains an interval.*

Remarks. It is known that a finite nth Peano derivative f_n possesses Darboux property, is of Baire class I, and is such that if it is bounded below or above in an interval then it is the ordinary nth derivative in that interval [8]. Since only these three properties of f'_{ap} are used in Theorem 1, the conclusion of Theorem 1 is valid if f'_{ap} is replaced by the finite nth Peano derivative f_n and f' is replaced by the ordinary nth derivative $f^{(n)}$.

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